# Edge solitons in the QHE

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## 1 Reduction of Chern-Simons

The Landau-Ginzburg theory of the Quantum Hall Effect [1] uses the Chern-Simons Lagrangian in $(2+1)$ dimensions,

$$\mathcal{L} = \frac{1}{4\kappa}\epsilon_{\mu\nu\rho}F_{\mu\nu}A_\rho + i\phi^\star D_t\phi - \frac{1}{2}|\vec{D}\phi|^2 - V(\phi), \tag{1}$$

where the scalar field $\phi$ is the order parameter and $A_\mu$ is the statistical gauge field; $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative. The constant $\kappa$ is interpreted as the Hall conductivity. The second-order field equations are not integrable [2]; they admit integrable reductions, though. The simplest of these is when time-dependence is eliminated; then, for a judicious choice of the self-interaction potential $V(\phi)$, the system admits finite-energy vortex solutions [3]. Here we focus our attention to another, space-like reduction [4]. Assuming independence from one spacelike coordinate and adding a suitable kinetic term yields in fact, after elimination of the gauge field using its equation of motion,

$$\mathcal{L} = i\phi^\star\partial_t\phi - \frac{1}{2}|(\partial_x - i\kappa^2\rho)\phi|^2 - V, \tag{2}$$

where $\rho = |\phi|^2$ is the particle density. This is the model proposed in Ref. [5] to describe the edge states in the QHE.

The field equations associated to (2) read

$$\begin{aligned} i\partial_t\phi &= -\tfrac{1}{2}(\partial_x - i\kappa^2\rho)^2\phi - \kappa^2 j\,\phi + \frac{\partial V}{\partial\phi^\star}, \\ j &= \tfrac{1}{2i}[\phi^\star(\partial_x - i\kappa^2\rho)\phi - \phi(\partial_x - i\kappa^2\rho)\phi)^\star]. \end{aligned} \tag{3}$$

Then the particle density and the current satisfy the continuity equation $\partial_t\rho + \partial_x j = 0$. Let us first assume that $V = 0$. Now the non-local transformation [6]

$$\psi = \left(\exp[-i\kappa^2\int^x \rho(y)dy]\right)\phi \tag{4}$$

takes (3) into the modified non-linear Schrödinger equation in which the density in the non-linearity has been replaced by the current,

$$
\begin{aligned}
i\partial_t \psi &= -\tfrac{1}{2}\partial_x^2\psi - 2\kappa^2 j\,\psi, \\
j &= \tfrac{1}{2i}[\psi^\star\partial_x\psi - \psi(\partial_x\psi)^\star].
\end{aligned}
\tag{5}
$$

## 2 The variable–coefficient NLS

Decomposing $\psi$ into module and phase, $\psi = \sqrt{\rho}\, e^{i\theta}$, yields (formally) the ordinary cubic NLS with variable coefficient,

$$
i\partial_t\psi = -\frac{1}{2}\partial_x^2\psi - F(t,x)|\psi|^2\psi, \tag{6}
$$

with $F(t,x) = 2\kappa^2\partial_x\theta$. Then Aglietti et al. [4] observe that, for $\theta = vx - \omega t$, Eq. (6) reduces to the usual non-linear Schrödinger equation with constant coefficient $F = 2\kappa^2 v$ which admits, for example, the travelling soliton solution

$$
\psi_s = \pm e^{i(vx-\omega t)}\sqrt{\frac{1}{2\kappa^2 v}}\frac{\alpha}{\cosh\alpha(x-vt)}, \qquad \alpha^2 = v^2 - 2\omega, \tag{7}
$$

is also consistent with the constraint. The non-linearity in (6) has to be attractive, $F > 0$; the solution (7) is therefore chiral, $v > 0$.

It is natural to ask whether the travelling soliton (7) can be generalized. Let us first study the variable-coefficient NLS (6) on its own. It has been shown [7] that this equation only passes the Painlevé test of Weiss, Tabor and Carnevale [8], when the coefficient of the non-linearity is

$$
F = (a + bt)^{-1}, \tag{8}
$$

where $a$ and $b$ constants. For $b = 0$, $F(t,x)$ is a constant and we recover the constant-coefficient NLS. For $b \neq 0$, the equation becomes explicitly time-dependent. Assuming, for simplicity, that $a = 0$ and $b = 1$, it reads

$$
i\partial_t\psi + \frac{1}{2}\partial_x^2\psi + \frac{1}{t}|\psi|^2\psi = 0. \tag{9}
$$

This equation can also be solved. Generalizing the usual travelling soliton, we find, for example, the 1-soliton

$$
\psi_0(t,x) = \frac{1}{\sqrt{t}}\frac{e^{i(x^2/4t - 1/2t)}}{\cosh\big[-x/t - x_0\big]}. \tag{10}
$$

It is worth pointing out that the steps followed in constructing (10) are essentially the same as those for the travelling soliton of the ordinary NLS — and this is not a pure coincidence. A short calculation shows in fact that

$$
\psi(t,x) = \frac{1}{\sqrt{t}}\exp\Big[\frac{ix^2}{4t}\Big]\,\Psi\big(-1/t, -x/t\big) \tag{11}
$$

satisfies the time-dependent equation (9) if and only if $U(t,x)$ solves Eqn. (6) with $F = 1$. Our soliton (10) comes in particular from the "standing soliton" $\Psi_s = \exp(it/2)\big(\cosh[x - x_0]\big)^{-1}$ solution of the NLS.

## 3 Non-relativistic conformal transformations

Where does the formula (10) come from ? The non-linear space-time transformation

$$D\ :\begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} -1/t \\ -x/t \end{pmatrix} \tag{12}$$

has already been met in a rather different context, namely in describing planetary motion when the gravitational "constant" changes inversely with time, as suggested by Dirac [9]. One shows in fact that $\vec{r}(t) = t\,\vec{r}^*(-1/t)$ describes planetary motion with Newton's "constant" varying as $G(t) = G_0/t$, whenever $\vec{r}^*(t)$ describes ordinary planetary motion, i.e. the one with a constant gravitational constant $G_0$ [10], [11]. The strange-looking transformation (12) is indeed related to the conformal structure of non-relativistic space-time [12]. It has been noticed in fact almost thirty years ago, that the "conformal" space-time transformations

$$\begin{array}{lll}
\begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} \delta^2 t \\ \delta\, x \end{pmatrix}, & 0 \neq \delta \in \mathbf{R} & \text{dilatations} \\[2ex]
\begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} \dfrac{t}{1-\kappa t} \\ \dfrac{x}{-\kappa t} \end{pmatrix}, & \kappa \in \mathbf{R} & \text{expansions} \\[2ex]
\begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} t+\epsilon \\ x \end{pmatrix}, & \epsilon \in \mathbf{R} & \text{time translations}
\end{array} \tag{13}$$

implemented on wave functions according to

$$\Psi(T,X) = \begin{cases} \delta^{1/2} u(t,x) \\ (1-\kappa t)^{1/2} \exp\Big[i \dfrac{\kappa x^2}{4(1-\kappa t)}\Big] \psi(t,x) \\ \psi(t,x) \end{cases} \tag{14}$$

permute the solutions of the free Schrödinger equation. In other words, they are symmetries of the free Schrödinger equation. The generators in (13) span in fact an $\mathrm{SL}(2,\mathbf{R})$ group. (A Dirac monopole, an Aharonov-Bohm vector potential and an inverse-square potential can also be included). The transformation $D$ in Eqn. (12) belongs to this symmetry group: it is in fact (i) a time translation with $\epsilon = 1$, (ii) followed by an expansion with $\kappa = 1$, (iii) followed by a second time-translation with $\epsilon = 1$. It is hence a symmetry for the free (linear) Schrödinger equation.

The cubic NLS with constant non-linearity is not more $SL(2,\mathbf{R})$ invariant: the transformation $D$ in (12) implemented as in Eq. (14) carries the cubic term into the time-dependent term $(1/t)|u|^2 u$, just like Newton's gravitational potential $G_0/r$ with $G_0 =$ const. is carried into the time-dependent Dirac expression $t^{-1}G_0/r$.

Similar arguments explain the integrability of other NLS type equations. For example, electromagnetic waves in a non-uniform medium propagate according to

$$i\partial_t\psi + \partial_x^2\psi + (-2\alpha x + 2|\psi|^2)\psi = 0, \tag{15}$$

which can again be solved by inverse scattering [13]. This is explained by observing that the potential term here can be eliminated by switching to a uniformly accelerated frame:

$$\begin{aligned} &\psi(t,x) = \exp\big[-i(2\alpha xt + \tfrac{4}{3}\alpha^2 t^3)\big]\Psi(T,X), \\ &T = t, \qquad X = x + 2\alpha t^2. \end{aligned} \tag{16}$$

Then $u(t,x)$ solves (15) whenever $U(T,X)$ solves the free equation.

The transformation (16) is again related to the structure of non-relativistic space-time. It can be shown in fact [11] that the (linear) Schrödinger equation

$$i\partial_t\psi + \partial_x^2\psi - V(t,x)\psi = 0 \tag{17}$$

can be brought into the free form by a space-time transformation if and only if the potential is $V(t,x) = \alpha(t)x \pm \frac{\omega^2(t)}{4}x^2$.For the uniform force field ($\omega = 0$) the required transformation is precisely (16).

For the oscillator potential ($\alpha = 0$), one can use rather Niederer's transformation [14]

$$\begin{aligned} &\psi(t,x) = (\cos\omega t)^{-1/2}\exp\big[-i\tfrac{\omega}{4}x^2\tan\omega t\big]\,\Psi(T,X), \\ &T = \frac{\tan\omega t}{\omega} \qquad X = \frac{x}{\cos\omega t}. \end{aligned} \tag{18}$$

Then $\psi$ satisfies the oscillator-equations iff $\Psi$ solves the free equation.

The Niederer transformation (18) leaves the inverse square potential invariant; this explains why the Calogero model in a harmonic background can be brought into the pure Calogero form [15]. Restoring the nonlinear term allows us to infer also that

$$i\partial_t\psi + \partial_x^2\psi + \Big(-\frac{\omega^2x^2}{4} + \frac{1}{\cos\omega t}\,|\psi|^2\Big)\psi = 0 \tag{19}$$

is integrable, and its solutions are obtained from those of the "free" NLS by the transformation (18). Let us mention that the covariance w. r. t. chronoprojective transformations was used before [16] for solving the NLS in oscillator and uniform–field backgrounds.

Now the constant-coefficient, damped, driven NLS,

$$i\partial_t\psi + \partial_x^2\psi + F|\psi|^2\psi = a(t,x)\psi + b(t,x), \tag{20}$$

passes the Painlevé test if

$$\begin{aligned} &a(t,x) = (\tfrac{1}{2}\partial_t\beta - \beta^2)x^2 + i\beta(t), + \alpha_1(t)x + \alpha_0(t), \\ &b(t,x) = 0 \end{aligned} \tag{21}$$

[17], i. e., precisely when the potential can be transformed away by our "non-relativistic conformal transformations".

# 4 An integrable extension

Unfortunately, the time-dependent travelling soliton (10) is inconsistent with the original equation (5), since its phase is quadratic in $x$ rather than linear, as required by consistency. The clue for finding integrable extensions is to observe that Eqn. (5) is in fact a Derivative Non-Linear Schrödinger equation (DNLS) [6]. Now the results of Clarkson and Cosgrove [18] say that the constant-coefficient equation

$$i\partial_t\psi + \frac{1}{2}\partial_x^2\psi + i(a\,\psi\psi^\star\partial_x\psi + b\,\psi^2\partial_x\psi^\star) + c\psi^3\psi^{\star 2} = 0 \tag{22}$$

is integrable iff

$$c = \frac{1}{2}b(2b-a). \tag{23}$$

In our case $a = -b = -\kappa^2$ and $c = 0$; Eq. (5) is therefore not integrable. However, adding a 6th–order potential to the Lagrange density i. e. considering rather

$$i\partial_t\psi + \frac{1}{2}\partial_x^2\psi + 2\kappa^2 j\,\psi + \frac{3}{2}\kappa^4|\psi|^4\psi = 0 \tag{24}$$

converts (5) into an integrable equation. Eq. (24) admits, e. g., the travelling wave solution $\psi = \sqrt{\rho}\,e^{i\theta}$, where

$$\rho = \frac{|v|}{2\kappa^2}\,\frac{1}{\sqrt{2}\cosh\left[v(x-\frac{v}{2}t)\right] + \operatorname{sign} v}, \qquad \theta = \frac{v}{2}x. \tag{25}$$

This can be checked by observing that for the Ansatz $\psi = f(x,t)e^{ivx/2}$ the modified NLS (24) again reduces to a constant-coefficient equation. Then the imaginary part of (24) requires that $f(x,t) = f(x-(v/2)t)$, while the real part can be integrated by the usual trick of multiplication by $f'$. The asymptotic conditions fix the integration constant to vanish, yielding a six-order non-linear equation, only containing even powers of $f$. Then, introducing $\rho = f^2$, we end up with the equation

$$(\rho')^2 - v^2\rho^2 + 4\kappa^2 v\rho^3 + 4\kappa^4\rho^4 = 0, \tag{26}$$

whose integration provides us with (25).

Another way of understanding how the integrability comes is to apply the non-local transformation (4) backwards, which carries (24) into a Derivative Non-linear Schrödinger equation of type II (DNLSII),

$$i\partial_t\phi + \frac{1}{2}\partial_x^2\phi + 2i\kappa^2\rho\,\partial_x\phi = 0, \tag{27}$$

which, consistently with Eq. (23), is integrable [19].

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